Problem 1.12

The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12),$$

where y is the distance (in miles) north, x the distance east, of South Hadley.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point 1 mile north and 1 mile east of South Hadley? In what direction is the slope steepest, at that point?

Solution

The extrema of a multidimensional function occur where the function's gradient is equal to zero.

$$\begin{aligned}
\nabla h &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}\right) h \\
&= \hat{\mathbf{x}}\frac{\partial h}{\partial x} + \hat{\mathbf{y}}\frac{\partial h}{\partial y} \\
&= \hat{\mathbf{x}}\frac{\partial}{\partial x}[10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)] \\
&\quad + \hat{\mathbf{y}}\frac{\partial}{\partial y}[10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)] \\
&= \hat{\mathbf{x}}[10(2y - 6x - 18)] \\
&\quad + \hat{\mathbf{y}}[10(2x - 8y + 28)]
\end{aligned}$$

The system of equations to solve is then

$$\left.\begin{aligned} 10(2y - 6x - 18) = 0 \\ 10(2x - 8y + 28) = 0 \end{aligned}\right\} \quad \Rightarrow \quad x = -2 \quad \text{and} \quad y = 3.$$

As a result, the extremum (or critical point) of $h(x, y)$ is $(-2, 3)$. Calculate the second derivatives of $h(x, y)$.

$$\begin{aligned}
\frac{\partial^2 h}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial}{\partial x}[10(2y - 6x - 18)] = 10(-6) = -60 \\
\frac{\partial^2 h}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}[10(2x - 8y + 28)] = 10(-8) = -80 \\
\frac{\partial^2 h}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial x}[10(2x - 8y + 28)] = 10(2) = 20
\end{aligned}$$

Apply the second derivative test to determine whether this extremum is a maximum, minimum, or saddle point.

$$D(-2, 3) = \begin{vmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{vmatrix}(-2, 3) = \frac{\partial^2 h}{\partial x^2}(-2, 3)\frac{\partial^2 h}{\partial y^2}(-2, 3) - \left[\frac{\partial^2 h}{\partial x \partial y}(-2, 3)\right]^2 = (-60)(-80) - (20)^2 = 4400$$

Since $D(-2,3) = 4400 > 0$ and $h_{xx}(-2,3) = -60 < 0$, the extremum at $(-2,3)$ is a maximum as expected. Therefore, the top of the hill is located 2 miles west and 3 miles north of South Hadley. Plug in $x = -2$ and $y = 3$ into $h(x,y)$ to find out how high the hill is.

$$h(-2,3) = 720$$

Therefore, the hill is 720 feet high. The direction of the steepest slope at any point is given by the gradient function.

$$\nabla h(x,y) = \hat{\mathbf{x}}[10(2y - 6x - 18)] + \hat{\mathbf{y}}[10(2x - 8y + 28)]$$

At a point 1 mile north and 1 mile east of South Hadley, the direction of the steepest slope is

$$\nabla h(1,1) = -220\hat{\mathbf{x}} + 220\hat{\mathbf{y}}.$$

Its magnitude tells how steep the slope is in feet per mile.

$$|\nabla h(1,1)| = \sqrt{(-220)^2 + (220)^2} = 220\sqrt{2} \approx 311$$